

# EMBEDDING DIMENSION AND CODIMENSION OF TENSOR PRODUCTS OF ALGEBRAS OVER A FIELD

S. BOUCHIBA (\*) AND S. KABBAJ (\*)

*To David Dobbs on the occasion of his 70th birthday*

**ABSTRACT.** Let  $k$  be a field. This paper investigates the embedding dimension and codimension of Noetherian local rings arising as localizations of tensor products of  $k$ -algebras. We use results and techniques from prime spectra and dimension theory to establish an analogue of the “special chain theorem” for the embedding dimension of tensor products, with effective consequence on the transfer or defect of regularity as exhibited by the (embedding) codimension given by  $\text{codim}(R) := \text{embdim}(R) - \dim(R)$ .

## 1. INTRODUCTION

Throughout, all rings are commutative with identity elements, ring homomorphisms are unital, and  $k$  stands for a field. The embedding dimension of a Noetherian local ring  $(R, \mathfrak{m})$ , denoted by  $\text{embdim}(R)$ , is the least number of generators of  $\mathfrak{m}$  or, equivalently, the dimension of  $\mathfrak{m}/\mathfrak{m}^2$  as an  $R/\mathfrak{m}$ -vector space. The ring  $R$  is regular if its Krull dimension and embedding dimensions coincide. The (embedding) codimension of  $R$  measures the defect of regularity of  $R$  and is given by the formula  $\text{codim}(R) := \text{embdim}(R) - \dim(R)$ . The concept of regularity was initially introduced by Krull and became prominent when Zariski showed that a local regular ring corresponds to a smooth point on an algebraic variety. Later, Serre proved that a ring is regular if and only if it has finite global dimension. This allowed to see that regularity is stable under localization and then the definition got globalized as follows: a Noetherian ring is regular if its localizations with respect to all prime ideals are regular. The ring  $R$  is a complete intersection if its  $\mathfrak{m}$ -completion is the quotient ring of a local regular ring modulo an ideal generated by a regular sequence;  $R$  is Gorenstein if its injective dimension is finite; and  $R$  is Cohen-Macaulay if the grade and height of  $\mathfrak{m}$  coincide. All these algebro-geometric notions are globalized by carrying over to localizations.

These concepts transfer to tensor products of algebras over a field under suitable assumptions. It has been proved that a Noetherian tensor product of algebras (over a field) inherits the notions of (locally) complete intersection ring, Gorenstein ring, and Cohen-Macaulay ring [7, 19, 33, 36]. In particular, a Noetherian tensor product of any two extension fields is a complete intersection ring. As to regularity and unlike the above notions, a Noetherian tensor product of two extension fields of

---

*Date:* January 20, 2017.

*2010 Mathematics Subject Classification.* 13H05, 13F20, 13B30, 13E05, 13D05, 14M05, 16E65.

*Key words and phrases.* Tensor product of  $k$ -algebras, regular ring, embedding dimension, Krull dimension, embedding codimension, separable extension.

(\*) Supported by KFUPM under DSR Research Grant # RG1212.

$k$  is not regular in general. In 1965, Grothendieck proved a positive result in case one of the two extension fields is a finitely generated separable extension [18]. Recently, we have investigated the possible transfer of regularity to tensor products of algebras over a field  $k$ . If  $A$  and  $B$  are two  $k$ -algebras such that  $A$  is geometrically regular; i.e.,  $A \otimes_k F$  is regular for every finite extension  $F$  of  $k$  (e.g.,  $A$  is a separable extension field over  $k$ ), we proved that  $A \otimes_k B$  is regular if and only if  $B$  is regular and  $A \otimes_k B$  is Noetherian [8, Lemma 2.1]. As a consequence, we established necessary and sufficient conditions for a Noetherian tensor product of two extension fields of  $k$  to inherit regularity under (pure in)separability conditions [8, Theorem 2.4]. Also, Majadas' relatively recent paper tackled questions of regularity and complete intersection of tensor products of commutative algebras via the homology theory of André and Quillen [25]. Finally, it is worthwhile recalling that tensor products of rings subject to the above concepts were recently used to broaden or delimit the context of validity of some homological conjectures; see for instance [20, 22]. Suitable background on regular, complete intersection, Gorenstein, and Cohen-Macaulay rings is [14, 18, 24, 26]. For a geometric treatment of these properties, we refer the reader to the excellent book of Eisenbud [15].

Throughout, given a ring  $R$ ,  $I$  an ideal of  $R$  and  $p$  a prime ideal of  $R$ , when no confusion is likely, we will denote by  $I_p$  the ideal  $IR_p$  of the local ring  $R_p$  and by  $\kappa_R(p)$  the residue field of  $R_p$ . One of the cornerstones of dimension theory of polynomial rings in several variables is the *special chain theorem*, which essentially asserts that the height of any prime ideal of the polynomial ring can always be realized via a special chain of prime ideals passing by the extension of its contraction over the basic ring; namely, if  $R$  is a Noetherian ring and  $P$  is a prime ideal of  $R[X_1, \dots, X_n]$  with  $p := P \cap R$ , then

$$\dim(R[X_1, \dots, X_n]_P) = \dim(R_p) + \dim\left(\kappa_R(p)[X_1, \dots, X_n]_{\frac{P_p}{pR_p[X_1, \dots, X_n]}}\right)$$

An analogue of this result for Noetherian tensor products, established in [7], states that, for any prime ideal  $P$  of  $A \otimes_k B$  with  $p := P \cap A$  and  $q := P \cap B$ , we have

$$\dim(A \otimes_k B)_P = \dim(A_p) + \dim\left((\kappa_A(p) \otimes_k B)_{\frac{P_p}{pA_p \otimes_k B}}\right)$$

which also comes in the following extended form

$$\dim(A \otimes_k B)_P = \dim(A_p) + \dim(B_q) + \dim\left((\kappa_A(p) \otimes_k \kappa_B(q))_{\frac{P(A_p \otimes_k B_q)}{pA_p \otimes_k B_q + A_p \otimes_k qB_q}}\right).$$

This paper investigates the embedding dimension of Noetherian local rings arising as localizations of tensor products of  $k$ -algebras. We use results and techniques from prime spectra and dimension theory to establish satisfactory analogues of the “special chain theorem” for the embedding dimension in various contexts of tensor products, with effective consequences on the transfer or defect of regularity as exhibited by the (embedding) codimension. The paper traverses four sections along with an introduction.

In Section 2, we introduce and study a new invariant which allows to correlate the embedding dimension of a Noetherian local ring  $B$  with the fibre ring  $B/\mathfrak{m}B$  of a local homomorphism  $f : A \rightarrow B$  of Noetherian local rings. This enables us to provide an analogue of the special chain theorem for the embedding dimension as

well as to generalize the known result that “if  $f$  is flat and  $A$  and  $B/\mathfrak{m}B$  are regular rings, then  $B$  is regular.”

Section 3 is devoted to the special case of polynomial rings which will be used in the investigation of tensor products. The main result (Theorem 3.1) states that, for a Noetherian ring  $R$  and  $X_1, \dots, X_n$  indeterminates over  $R$ , for any prime ideal  $P$  of  $R[X_1, \dots, X_n]$  with  $p := P \cap R$ , we have:

$$\begin{aligned} \text{embdim}(R[X_1, \dots, X_n]_P) &= \text{embdim}(R_p) + \text{ht}\left(\frac{P}{p[X_1, \dots, X_n]}\right) \\ &= \text{embdim}(R_p) + \text{embdim}\left(\kappa_R(p)[X_1, \dots, X_n]_{\frac{p_p}{pR_p[X_1, \dots, X_n]}}\right) \end{aligned}$$

Then, Corollary 3.2 asserts that

$$\text{codim}(R[X_1, \dots, X_n]_P) = \text{codim}(R_p)$$

and recovers a well-known result on the transfer of regularity to polynomial rings; i.e.,  $R[X_1, \dots, X_n]$  is regular if and only if so is  $R$  (this result was initially proved via Serre’s result on finite global dimension and Hilbert Theorem on syzygies). Then Corollary 3.3 characterizes regularity in general settings of localizations of polynomial rings and, in the particular cases of Nagata rings and Serre conjecture rings, it states that  $R(X_1, \dots, X_n)$  is regular if and only if  $R\langle X_1, \dots, X_n \rangle$  is regular if and only if  $R$  is regular.

Let  $A$  and  $B$  be two  $k$ -algebras such that  $A \otimes_k B$  is Noetherian and let  $P$  be a prime ideal of  $A \otimes_k B$  with  $p := P \cap A$  and  $q := P \cap B$ . Due to known behavior of tensor products of  $k$ -algebras subject to regularity (cf. [8, 18, 19, 33, 36]), Section 4 investigates the case when  $A$  (or  $B$ ) is a separable (not necessarily algebraic) extension field of  $k$ . The main result (Theorem 4.2) asserts that, if  $K$  is a separable extension field of  $k$ , then

$$\text{embdim}(K \otimes_k A)_P = \text{embdim}(A_p) + \text{embdim}\left(\left(K \otimes_k \kappa_A(p)\right)_{\frac{p_p}{K \otimes_k p A_p}}\right).$$

In particular, if  $K$  is separable algebraic over  $k$ , then

$$\text{embdim}(K \otimes_k A)_P = \text{embdim}(A_p).$$

Then, Corollary 4.5 asserts that

$$\text{codim}(K \otimes_k A)_P = \text{codim}(A_p)$$

and hence  $K \otimes_k A$  is regular if and only if so is  $A$ . This recovers Grothendieck’s result on the transfer of regularity to tensor products issued from finite extension fields [18, Lemma 6.7.4.1].

Section 5 examines the more general case of tensor products of  $k$ -algebras with separable residue fields. The main theorem (Theorem 5.1) states that if  $\kappa_B(q)$  is a separable extension field of  $k$ , then

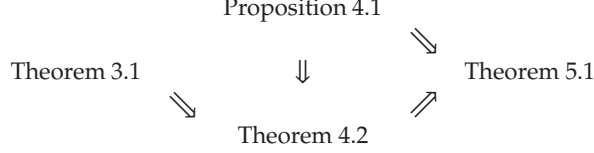
$$\begin{aligned} \text{embdim}(A \otimes_k B)_P &= \text{embdim}(A_p) + \text{embdim}(B_q) \\ &\quad + \text{embdim}\left(\left(\kappa_A(p) \otimes_k \kappa_B(q)\right)_{\frac{p(A_p \otimes_k B_q)}{p A_p \otimes_k B_q + A_p \otimes_k q B_q}}\right) \end{aligned}$$

Then, Corollary 5.2 contends that

$$\text{codim}(A \otimes_k B)_P = \text{codim}(A_p) + \text{codim}(B_q)$$

recovering known results on the transfer of regularity to tensor products over perfect fields [33, Theorem 6(c)] and, more generally, to tensor products issued from residually separable extension fields [8, Theorem 2.11].

The four aforementioned main results are connected as follows:



Of relevance to this study is Bouchiba, Conde-Lago, and Majadas' recent preprint [4] where the authors prove some of our results via the homology theory of André and Quillen. In the current paper, we offer direct and self-contained proofs using techniques and basic results from commutative ring theory. Early and recent developments on prime spectra and dimension theory are to be found in [3, 5, 6, 7, 29, 30, 31, 34, 35] for the special case of tensor products of  $k$ -algebras, and in [1, 11, 17, 22, 24, 26, 27] for the general case. Any unreferenced material is standard, as in [24, 26].

## 2. EMBEDDING DIMENSION OF NOETHERIAN LOCAL RINGS

In this section, we discuss the relationship between the embedding dimensions of Noetherian local rings connected by a local ring homomorphism. To this purpose, we introduce a new invariant  $\mu$  which allows to relate the embedding dimension of a local ring to that of its fibre ring.

Throughout, let  $(A, \mathfrak{m}, K)$  and  $(B, \mathfrak{n}, L)$  be local Noetherian rings,  $f : A \rightarrow B$  a local homomorphism (i.e.,  $\mathfrak{m}B := f(\mathfrak{m})B \subseteq \mathfrak{n}$ ), and  $I$  a proper ideal of  $A$ . Let

$$\mu_A(I) := \dim_K \left( \frac{I + \mathfrak{m}^2}{\mathfrak{m}^2} \right).$$

Note that  $\mu_A(I)$  equals the maximal number of elements of  $I$  which are part of a minimal basis of  $\mathfrak{m}$ ; so that  $0 \leq \mu_A(I) \leq \text{embdim}(A)$  and  $\mu_A(\mathfrak{m}) = \text{embdim}(A)$ . Next, let  $\mu_B^f(I)$  denote the maximal number of elements of  $IB := f(I)B$  which are part of a minimal basis of  $\mathfrak{n}$ ; that is,

$$\mu_B^f(I) := \mu_B(IB) = \dim_L \left( \frac{IB + \mathfrak{n}^2}{\mathfrak{n}^2} \right).$$

It is easily seen that if  $x_1, \dots, x_r$  are elements of  $\mathfrak{m}$  such that  $f(x_1), \dots, f(x_r)$  are part of a minimal basis of  $\mathfrak{n}$ , then  $x_1, \dots, x_r$  are part of a minimal basis of  $\mathfrak{m}$  as well. That is,  $0 \leq \mu_B^f(I) \leq \mu_A(I)$ . Moreover, if  $J$  is a proper ideal of  $B$  and  $\pi : B \rightarrow B/J$  is the canonical surjection, then the natural linear map of  $L$ -vector spaces  $\frac{IB + \mathfrak{n}^2}{\mathfrak{n}^2} \rightarrow \frac{IB + \mathfrak{n}^2 + J}{\mathfrak{n}^2 + J}$  yields

$$\mu_{B/J}^{\pi \circ f}(I) \leq \mu_B^f(I).$$

**Proposition 2.1.** *Under the above notation, we have:*

$$\text{embdim}(B) = \mu_B^f(I) + \text{embdim}(B/IB).$$

*In particular,*

$$\text{embdim}(A) = \mu_A(I) + \text{embdim}(A/I).$$

*Proof.* The first statement follows easily from the following exact sequence of  $L$ -vector spaces

$$0 \longrightarrow \frac{IB + n^2}{n^2} \longrightarrow \frac{n}{n^2} \longrightarrow \frac{n}{IB + n^2} = \frac{n/IB}{(n/IB)^2} \longrightarrow 0.$$

The second statement holds since  $\mu_A(I) = \mu_A^{\text{id}_A}(I)$ .  $\square$

Recall that, under the above notation, the following inequality always holds:  $\dim(B) \leq \dim(A) + \dim(B/\mathfrak{m}B)$ . The first corollary provides an analogue for the embedding dimension.

**Corollary 2.2.** *Under the above notation, we have:*

$$\text{embdim}(B) \leq \text{embdim}(A) - \text{embdim}(A/I) + \text{embdim}(B/IB).$$

*In particular,*

$$\text{embdim}(B) \leq \text{embdim}(A) + \text{embdim}(B/\mathfrak{m}B).$$

It is well known that if  $f$  is flat and both  $A$  and  $B/\mathfrak{m}B$  are regular, then  $B$  is regular. The second corollary generalizes this result to homomorphisms subject to going-down. Recall that a ring homomorphism  $h : R \rightarrow S$  satisfies going-down (henceforth abbreviated GD) if for any pair  $p \subseteq q$  in  $\text{Spec}(R)$  such that there exists  $Q \in \text{Spec}(S)$  lying over  $q$ , then there exists  $P \in \text{Spec}(S)$  lying over  $p$  with  $P \subseteq Q$ . Any flat ring homomorphism satisfies GD.

**Corollary 2.3.** *Under the above notation, assume that  $f$  satisfies GD. Then:*

- (a)  $\text{codim}(B) = (\mu_B^f(\mathfrak{m}) - \dim(A)) + \text{codim}(B/\mathfrak{m}B)$ .
- (b)  $\text{codim}(B) + (\text{embdim}(A) - \mu_B^f(\mathfrak{m})) = \text{codim}(A) + \text{codim}(B/\mathfrak{m}B)$ .
- (c)  $B$  is regular and  $\mu_B^f(\mathfrak{m}) = \text{embdim}(A) \iff A$  and  $B/\mathfrak{m}B$  are regular.

*Proof.* The proof is straightforward via a combination of Proposition 2.1 and [26, Theorem 15.1].  $\square$

**Corollary 2.4.** *Under the above notation, assume that  $f$  satisfies GD. Then:*

- (a)  $\text{codim}(B) \leq \text{codim}(A) + \text{codim}(B/\mathfrak{m}B)$ .
- (b) *If  $B/\mathfrak{m}B$  is regular, then  $\text{codim}(B) \leq \text{codim}(A)$ .*

*Proof.* The proof is direct via a combination of Corollary 2.2 and the known fact that  $\dim(B) = \dim(A) + \dim(B/\mathfrak{m}B)$ .  $\square$

### 3. EMBEDDING DIMENSION AND CODIMENSION OF POLYNOMIAL RINGS

This section is devoted to the special case of polynomial rings which will be used, later, for the investigation of tensor products. The main result of this section (Theorem 3.1) settles a formula for the embedding dimension for the localizations of polynomial rings over Noetherian rings. It recovers (via Corollary 3.2) a well-known result on the transfer of regularity to polynomial rings; that is,  $R[X_1, \dots, X_n]$  is regular if and only if so is  $R$ . Moreover, Theorem 3.1 leads to investigate the regularity of two famous localizations of polynomial rings in several variables; namely, the Nagata ring  $R(X_1, X_2, \dots, X_n)$  and Serre conjecture ring  $R\langle X_1, X_2, \dots, X_n \rangle$ . We show that the regularity of these two constructions is entirely characterized by the regularity of  $R$  (Corollary 3.3).

Recall that one of the cornerstones of dimension theory of polynomial rings in several variables is *the special chain theorem*, which essentially asserts that the height of any prime ideal  $P$  of  $R[X_1, \dots, X_n]$  can always be realized via a special chain of prime ideals passing by the extension  $(P \cap R)[X_1, \dots, X_n]$ . This result was first proved by Jaffard in [22] and, later, Brewer, Heinzer, Montgomery and Rutter reformulated it in the following simple way ([12, Theorem 1]): Let  $P$  be a prime ideal of  $R[X_1, \dots, X_n]$  with  $p := P \cap R$ . Then  $\text{ht}(P) = \text{ht}(p[X_1, \dots, X_n]) + \text{ht}\left(\frac{P}{p[X_1, \dots, X_n]}\right)$ . In a Noetherian setting, this formula becomes:

$$\begin{aligned} \dim(R[X_1, \dots, X_n]_P) &= \dim(R_p) + \text{ht}\left(\frac{P}{p[X_1, \dots, X_n]}\right) \\ &= \dim(R_p) + \dim\left(\kappa_R(p)[X_1, \dots, X_n]_{\frac{p_p}{pR_p[X_1, \dots, X_n]}}\right) \end{aligned} \quad (1)$$

where the second equality holds on account of the basic fact  $\frac{P}{p[X_1, \dots, X_n]} \cap \frac{R}{p} = 0$ . The main result of this section (Theorem 3.1) features a “special chain theorem” for the embedding dimension with effective consequence on the codimension.

**Theorem 3.1.** *Let  $R$  be a Noetherian ring and  $X_1, \dots, X_n$  be indeterminates over  $R$ . Let  $P$  be a prime ideal of  $R[X_1, \dots, X_n]$  with  $p := P \cap R$ . Then:*

$$\begin{aligned} \text{embdim}(R[X_1, \dots, X_n]_P) &= \text{embdim}(R_p) + \text{ht}\left(\frac{P}{p[X_1, \dots, X_n]}\right) \\ &= \text{embdim}(R_p) + \text{embdim}\left(\kappa_R(p)[X_1, \dots, X_n]_{\frac{p_p}{pR_p[X_1, \dots, X_n]}}\right) \end{aligned}$$

*Proof.* We use induction on  $n$ . Assume  $n = 1$  and let  $P$  be a prime ideal of  $R[X]$  with  $p := P \cap R$  and  $r := \text{embdim}(R_p)$ . Then  $p_p = (a_1, \dots, a_r)R_p$  for some  $a_1, \dots, a_r \in p$ . We envisage two cases; namely, either  $P$  is an extension of  $p$  or an upper to  $p$ . For both cases, we will use induction on  $r$ .

**Case 1:**  $P$  is an extension of  $p$  (i.e.,  $P = pR[X]$ ). We prove that  $\text{embdim}(R[X]_P) = r$ . Indeed, we have  $P_P = pR_p[X]_{pR_p[X]} = (a_1, \dots, a_r)R_p[X]_{pR_p[X]} = (a_1, \dots, a_r)R[X]_P$ . So, obviously, if  $p_p = (0)$ , then  $P_P = 0$ . Next, we may assume  $r \geq 1$ . One can easily check that the canonical ring homomorphism  $\varphi : R_p \rightarrow R[X]_P$  is injective with  $\varphi(p_p) \subseteq P_P$ . This forces  $\text{embdim}(R[X]_P) \geq 1$ . Hence, there exists  $j \in \{1, \dots, n\}$ , say  $j = 1$ , such that  $a := a_1 \in p$  with  $\frac{a}{1} \in P_P \setminus P_P^2$  and, a fortiori,  $\frac{a}{1} \in p_p \setminus p_p^2$ . By [24, Theorem 159], we get

$$\begin{cases} \text{embdim}(R[X]_P) = 1 + \text{embdim}\left(\frac{R}{(a)}[X]_{\frac{p}{aR[X]}}\right) \\ \text{embdim}(R_p) = 1 + \text{embdim}\left(\left(\frac{R}{(a)}\right)_{\frac{p}{(a)}}\right) \end{cases} \quad (2)$$

Therefore  $\text{embdim}\left(\left(\frac{R}{(a)}\right)_{\frac{p}{(a)}}\right) = r - 1$  and then, by induction on  $r$ , we obtain

$$\text{embdim}\left(\frac{R}{(a)}[X]_{\frac{p}{aR[X]}}\right) = \text{embdim}\left(\left(\frac{R}{(a)}\right)_{\frac{p}{(a)}}\right). \quad (3)$$

A combination of (2) and (3) leads to  $\text{embdim}(R[X]_P) = r$ , as desired.

**Case 2:**  $P$  is an upper to  $p$  (i.e.,  $P \neq pR[X]$ ). We prove that  $\text{embdim}(R[X]_P) = r + 1$ . Note that  $PR_p[X]$  is also an upper to  $p_p$  and then there exists a (monic) polynomial  $f \in R[X]$  such that  $\frac{f}{1}$  is irreducible in  $\kappa_R(p)[X]$  and  $PR_p[X] = pR_p[X] + fR_p[X]$ . Notice that  $pR[X] + fR[X] \subseteq P$  and we have

$$\begin{aligned} P_P &= PR_p[X]_{PR_p[X]} = (pR_p[X] + fR_p[X])_{PR_p[X]} \\ &= (p[X] + fR[X])_{R_p[X]_{PR_p[X]}} = (p[X] + fR[X])_P \\ &= p[X]_P + fR[X]_P = (a_1, \dots, a_r, f)R[X]_P. \end{aligned}$$

Assume  $r = 0$ . Then  $P$  is an upper to zero with  $P_P = fR[X]_P$ . So that  $\text{embdim}(R[X]_P) \leq 1$ . Further, by the principal ideal theorem [24, Theorem 152], we have

$$\text{embdim}(R[X]_P) \geq \dim(R[X]_P) = \text{ht}(P) = 1.$$

It follows that  $\text{embdim}(R[X]_P) = 1$ , as desired.

Next, assume  $r \geq 1$ . We claim that  $pR[X]_P \not\subseteq P_P^2$ . Deny and suppose that  $pR[X]_P \subseteq P_P^2$ . This assumption combined with the fact  $P_P = p[X]_P + fR[X]_P$  yields  $\frac{P_P}{P_P^2} = \frac{p[X]_P}{P_P^2} + \frac{fR[X]_P}{P_P^2}$  as  $R[X]_P$ -modules and hence  $P_P = fR[X]_P$  by [24, Theorem 158]. Next, let  $a \in p$ . Then, as  $\frac{a}{1} \in P_P = fR[X]_P$ , there exist  $g \in R[X]$  and  $s, t \in R[X] \setminus P$  such that  $t(sa - fg) = 0$ . So that  $tf g \in p[X]$ , whence  $tg \in p[X] \subset P$  as  $f \notin p[X]$ . It follows that  $t sa = tf g \in P^2$  and thus  $\frac{a}{1} \in P_P^2 = f^2 R[X]_P$ . We iterate the same process to get  $\frac{a}{1} \in P_P^n = f^n R[X]_P$  for each integer  $n \geq 1$ . Since  $R[X]_P$  is a Noetherian local ring,  $\bigcap P_P^n = (0)$  and thus  $\frac{a}{1} = 0$  in  $R[X]_P$ . By the canonical injective homomorphism  $R_p \hookrightarrow R[X]_P$ ,  $\frac{a}{1} = 0$  in  $R_p$ . Thus  $p_p = (0)$ , the desired contradiction.

Consequently,  $pR[X]_P = (a_1, \dots, a_r)R[X]_P \not\subseteq P_P^2$ . So, there exists  $j \in \{1, \dots, r\}$ , say  $j = 1$ , such that  $a := a_1 \in P_P \setminus P_P^2$  and, a fortiori,  $a \in p_p \setminus p_p^2$ . Similar arguments as in Case 1 lead to the same two formulas displayed in (2). Therefore  $\text{embdim}\left(\left(\frac{R}{(a)}\right)_{\frac{p}{(a)}}\right) = r - 1$  and then, by induction on  $r$ , we obtain

$$\text{embdim}\left(\frac{R}{(a)}[X]_{\frac{p}{aR[X]}}\right) = 1 + \text{embdim}\left(\left(\frac{R}{(a)}\right)_{\frac{p}{(a)}}\right). \quad (4)$$

A combination of (2) and (4) leads to  $\text{embdim}(R[X]_P) = r + 1$ , as desired.

Now, assume that  $n \geq 2$  and set  $R[k] := R[X_1, \dots, X_k]$  and  $p[k] = p[X_1, \dots, X_k]$  for  $k := 1, \dots, n$ . Let  $P' := P \cap R[n-1]$ . We prove that  $\text{embdim}(R[n]_P) = r + \text{ht}\left(\frac{P}{p[n]}\right)$ . Indeed, by virtue of the case  $n = 1$ , we have

$$\text{embdim}(R[n]_P) = \text{embdim}(R[n-1]_{P'}) + \text{ht}\left(\frac{P}{P'[X_n]}\right). \quad (5)$$

Moreover, by induction hypothesis, we get

$$\text{embdim}(R[n-1]_{P'}) = r + \text{ht}\left(\frac{P'}{p[n-1]}\right). \quad (6)$$



Note that the prime ideals  $\frac{P'[X_n]}{p[n]}$  and  $\frac{P}{p[n]}$  both survive in  $\kappa_R(p)[n]$ , respectively. Hence, as  $\kappa_R(p)[n]$  is catenarian and  $(R/p)[n-1]$  is Noetherian, we obtain

$$\text{ht}\left(\frac{P}{p[n]}\right) = \text{ht}\left(\frac{P'[X_n]}{p[n]}\right) + \text{ht}\left(\frac{P}{P'[X_n]}\right) = \text{ht}\left(\frac{P'}{p[n-1]}\right) + \text{ht}\left(\frac{P}{P'[X_n]}\right). \quad (7)$$

Further, the fact that  $\kappa_R(p)[X_1, \dots, X_n]$  is regular yield

$$\text{ht}\left(\frac{P}{p[X_1, \dots, X_n]}\right) = \text{embdim}\left(\kappa_R(p)[X_1, \dots, X_n]_{\frac{p_p}{pR_p[X_1, \dots, X_n]}}\right). \quad (8)$$

So (5), (6), (7), and (8) lead to the conclusion, completing the proof of the theorem.  $\square$

As a first application of Theorem 3.1, we get the next corollary on the (embedding) codimension. In particular, it recovers a well-known result on the transfer of regularity to polynomial rings (initially proved via Serre's result on finite global dimension and Hilbert Theorem on syzygies [28, Theorem 8.37]. See also [24, Theorem 171]).

**Corollary 3.2.** *Let  $R$  be a Noetherian ring and  $X_1, \dots, X_n$  be indeterminates over  $R$ . Let  $P$  be a prime ideal of  $R[X_1, \dots, X_n]$  with  $p := P \cap R$ . Then:*

$$\text{codim}(R[X_1, \dots, X_n]_P) = \text{codim}(R_p).$$

*In particular,  $R[X_1, \dots, X_n]$  is regular if and only if  $R$  is regular.*

Theorem 3.1 allows us to characterize the regularity for two famous localizations of polynomial rings; namely, Nagata rings and Serre conjecture rings. Let  $R$  be a ring and  $X, X_1, \dots, X_n$  indeterminates over  $R$ . Recall that  $R(X_1, \dots, X_n) = S^{-1}R[X_1, \dots, X_n]$  is the Nagata ring, where  $S$  is the multiplicative set of  $R[X_1, \dots, X_n]$  consisting of the polynomials whose coefficients generate  $R$ . Let  $R\langle X \rangle := U^{-1}R[X]$ , where  $U$  is the multiplicative set of monic polynomials in  $R[X]$ , and  $R\langle X_1, \dots, X_n \rangle := R\langle X_1, \dots, X_{n-1} \rangle\langle X_n \rangle$ . Then  $R\langle X_1, \dots, X_n \rangle$  is called the Serre conjecture ring and is a localization of  $R[X_1, \dots, X_n]$ .

**Corollary 3.3.** *Let  $R$  be a Noetherian ring and  $X_1, \dots, X_n$  indeterminates over  $R$ . Let  $S$  be a multiplicative subset of  $R[X_1, \dots, X_n]$ . Then:*

- (a)  $S^{-1}R[X_1, \dots, X_n]$  is regular if and only if  $R_p$  is regular for each prime ideal  $p$  of  $R$  such that  $p[X_1, \dots, X_n] \cap S = \emptyset$ .
- (b) In particular,  $R(X_1, \dots, X_n)$  is regular if and only if  $R\langle X_1, \dots, X_n \rangle$  is regular if and only if  $R[X_1, \dots, X_n]$  is regular if and only if  $R$  is regular.

*Proof.* (a) Let  $Q = S^{-1}P$  be a prime ideal of  $S^{-1}R[X_1, \dots, X_n]$ , where  $P$  is the inverse image of  $Q$  by the canonical homomorphism  $R[X_1, \dots, X_n] \rightarrow S^{-1}R[X_1, \dots, X_n]$  and let  $p := P \cap R$ . Notice that  $S^{-1}R[X_1, \dots, X_n]_Q \cong R[X_1, \dots, X_n]_P$  and

$\frac{Q}{S^{-1}p[X_1, \dots, X_n]} \cong \bar{S}^{-1} \frac{P}{p[X_1, \dots, X_n]}$  where  $\bar{S}$  denotes the image of  $S$  via the natural homomorphism  $R[X_1, \dots, X_n] \rightarrow \frac{R}{p}[X_1, \dots, X_n]$ . Therefore, by (1), we obtain

$$\dim(S^{-1}R[X_1, \dots, X_n]_Q) = \dim(R[X_1, \dots, X_n]_P) = \dim(R_p) + \text{ht}\left(\frac{Q}{S^{-1}p[X_1, \dots, X_n]}\right) \quad (9)$$



and, by Theorem 3.1, we have

$$\begin{aligned} \text{embdim}(S^{-1}R[X_1, \dots, X_n]_Q) &= \text{embdim}(R[X_1, \dots, X_n]_P) \\ &= \text{embdim}(R_P) + \text{ht}\left(\frac{Q}{S^{-1}P[X_1, \dots, X_n]}\right). \end{aligned} \quad (10)$$

Now, observe that the set  $\{Q \cap R \mid Q \text{ is a prime ideal of } S^{-1}R[X_1, \dots, X_n]\}$  is equal to the set  $\{p \mid p \text{ is a prime ideal of } R \text{ such that } p[X_1, \dots, X_n] \cap S = \emptyset\}$ . Therefore, (9) and (10) lead to the conclusion.

(b) Combine (a) with the fact that the extension of any prime ideal of  $R$  to  $R[X_1, \dots, X_n]$  does not meet the multiplicative sets related to the rings  $R(X_1, \dots, X_n)$  and  $R\langle X_1, \dots, X_n \rangle$ .  $\square$

#### 4. EMBEDDING DIMENSION AND CODIMENSION OF TENSOR PRODUCTS ISSUED FROM SEPARABLE EXTENSION FIELDS

This section establishes an analogue of the “special chain theorem” for the embedding dimension of Noetherian tensor products issued from separable extension fields, with effective consequences on the transfer or defect of regularity. Namely, due to known behavior of a tensor product  $A \otimes_k B$  of two  $k$ -algebras subject to regularity (cf. [8, 18, 19, 26, 33, 36]), we will investigate the case where  $A$  or  $B$  is a separable (not necessarily algebraic) extension field of  $k$ .

Throughout, let  $A$  and  $B$  be two  $k$ -algebras such that  $A \otimes_k B$  is Noetherian and let  $P$  be a prime ideal of  $A \otimes_k B$  with  $p := P \cap A$  and  $q := P \cap B$ . Recall that  $A$  and  $B$  are Noetherian too; and the converse is not true, in general, even if  $A = B$  is an extension field of  $k$  (cf. [16, Corollary 3.6] or [34, Theorem 11]). We assume familiarity with the natural isomorphisms for tensor products and their localizations as in [9, 10, 28]. In particular, we identify  $A$  and  $B$  with their respective images in  $A \otimes_k B$  and we have  $\frac{A \otimes_k B}{p \otimes_k B + A \otimes_k q} \cong \frac{A}{p} \otimes_k \frac{B}{q}$  and  $A_p \otimes_k B_q \cong S^{-1}(A \otimes_k B)$  where  $S := \{s \otimes t \mid s \in A \setminus p, t \in B \setminus q\}$ . Throughout this and next sections, we adopt the following simplified notation for the invariant  $\mu$ :

$$\mu_P(pA_p) := \mu_{(A \otimes_k B)_P}^i(pA_p) \text{ and } \mu_P(qA_q) := \mu_{(A \otimes_k B)_P}^j(qB_q)$$

where  $i : A_p \rightarrow (A \otimes_k B)_P$  and  $j : B_q \rightarrow (A \otimes_k B)_P$  are the canonical (local flat) ring homomorphisms.

Recall that  $A \otimes_k B$  is Cohen-Macaulay (resp., Gorenstein, locally complete intersection) if and only if so are  $A$  and the fibre rings  $\kappa_A(p) \otimes_k B$  (for each prime ideal  $p$  of  $A$ ) [7, 33]. Also if  $A$  and the fibre rings  $\kappa_A(p) \otimes_k B$  are regular then so is  $A \otimes_k B$  [26, Theorem 23.7(ii)]. However, the converse does not hold in general; precisely, if  $A \otimes_k B$  is regular then so is  $A$  [26, Theorem 23.7(i)] but the fibre rings are not necessarily regular (see [8, Example 2.12(iii)]).

From [7, Proposition 2.3] and its proof, recall an analogue of the special chain theorem (recorded in (1)) for the tensor products which correlates the dimension of  $(A \otimes_k B)_P$  to the dimension of its fibre rings; namely,

$$\begin{aligned} \dim(A \otimes_k B)_P &= \dim(A_p) + \text{ht}\left(\frac{P}{p \otimes_k B}\right) \\ &= \dim(A_p) + \dim\left(\left(\kappa_A(p) \otimes_k B\right)_{\frac{P_p}{pA_p \otimes_k B}}\right) \end{aligned} \quad (11)$$

Our first result reformulates Proposition 2.1 and thus gives an analogue of the special chain theorem for the embedding dimension in the context of tensor products of algebras over a field.

**Proposition 4.1.** *Let  $A$  and  $B$  be two  $k$ -algebras such that  $A \otimes_k B$  is Noetherian and let  $P$  be a prime ideal of  $A \otimes_k B$  with  $p := P \cap A$  and  $q := P \cap B$ . Then:*

- (a)  $\text{embdim}(A \otimes_k B)_P = \mu_P(pA_p) + \text{embdim}\left(\left(\kappa_A(p) \otimes_k B\right)_{\frac{P_p}{pA_p \otimes_k B}}\right).$
- (b)  $\text{codim}(A \otimes_k B)_P + (\text{embdim}(A_p) - \mu_P(pA_p)) = \text{codim}(A_p) + \text{codim}\left(\left(\kappa_A(p) \otimes_k B\right)_{\frac{P_p}{pA_p \otimes_k B}}\right).$
- (c)  $(A \otimes_k B)_P$  is regular and  $\mu_P(pA_p) = \text{embdim}(A_p)$  if and only if both  $A_p$  and  $\left(\kappa_A(p) \otimes_k B\right)_{\frac{P_p}{pA_p \otimes_k B}}$  are regular.

Recall that an extended form of the special chain theorem [7] states that

$$\dim(A \otimes_k B)_P = \dim(A_p) + \dim(B_q) + \dim\left(\left(\kappa_A(p) \otimes_k \kappa_B(q)\right)_{\frac{P(A_p \otimes_k B_q)}{pA_p \otimes_k B_q + A_p \otimes_k qB_q}}\right).$$

In this vein, notice that, via Proposition 4.1(a), we always have the following inequalities:

$$\begin{aligned} \text{embdim}(A \otimes_k B)_P &\leq \text{embdim}(A_p) + \text{embdim}\left(\left(\kappa_A(p) \otimes_k B\right)_{\frac{P_p}{pA_p \otimes_k B}}\right) \\ &\leq \text{embdim}(A_p) + \text{embdim}(B_q) \\ &\quad + \text{embdim}\left(\left(\kappa_A(p) \otimes_k \kappa_B(q)\right)_{\frac{P(A_p \otimes_k B_q)}{pA_p \otimes_k B_q + A_p \otimes_k qB_q}}\right). \end{aligned}$$

Let us state the main theorem of this section.

**Theorem 4.2.** *Let  $K$  be a separable extension field of  $k$  and  $A$  a  $k$ -algebra such that  $K \otimes_k A$  is Noetherian. Let  $P$  be a prime ideal of  $K \otimes_k A$  with  $p := P \cap A$ . Then:*

$$\begin{aligned} \text{embdim}(K \otimes_k A)_P &= \text{embdim}(A_p) + \text{ht}\left(\frac{P}{K \otimes_k p}\right) \\ &= \text{embdim}(A_p) + \text{embdim}\left(\left(K \otimes_k \kappa_A(p)\right)_{\frac{P_p}{K \otimes_k pA_p}}\right) \end{aligned}$$

*If, in addition,  $K$  is algebraic over  $k$ , then  $\text{embdim}(K \otimes_k A)_P = \text{embdim}(A_p)$ .*

The proof of this theorem requires the following two preparatory lemmas; the first of which determines a formula for the embedding dimension of the tensor product of two  $k$ -algebras  $A$  and  $B$  localized at a special prime ideal  $P$  with no restrictive conditions on  $A$  or  $B$ .

**Lemma 4.3.** *Let  $A$  and  $B$  be two  $k$ -algebras such that  $A \otimes_k B$  is Noetherian and let  $P$  be a prime ideal of  $A \otimes_k B$  with  $p := P \cap A$  and  $q := P \cap B$ . Assume that  $P_P = (p \otimes_k B + A \otimes_k q)_P$ . Then:*

- (a)  $\mu_P(pA_p) = \text{embdim}(A_p)$  and  $\mu_P(qB_q) = \text{embdim}(B_q)$ .
- (b)  $\text{embdim}(A \otimes_k B)_P = \text{embdim}(A_p) + \text{embdim}(B_q)$ .

*Proof.* We proceed through two steps.

**Step 1.** Assume that  $K := B$  is an extension field of  $k$ . Then  $q = (0)$  and  $P_p = p_p(A_p \otimes_k K)_{P_p}$ . Let  $n := \text{embdim}(A_p)$  and let  $a_1, \dots, a_n$  be elements of  $p$  such that  $p_p = \left(\frac{a_1}{1}, \dots, \frac{a_n}{1}\right)A_p$ . Our argument uses induction on  $n$ . If  $n = 0$ , then  $A_p$  is a field and  $p_p = (0)$ ; hence  $P_p = (0)$ , whence  $\text{embdim}(A \otimes_k K)_P = 0$ , as desired. Next, suppose  $n \geq 1$ . We have  $P_p = \left(\frac{a_1}{1}, \dots, \frac{a_n}{1}\right)(A \otimes_k K)_P$ . If  $\text{embdim}(A \otimes_k K)_P = 0$ ,  $(A \otimes_k K)_P$  is regular and so is  $A_p$  by [26, Theorem 23.7(i)]. Hence,  $n = \dim(A_p) = 0$  by (11). Absurd. So, necessarily,  $\text{embdim}(A \otimes_k K)_P \geq 1$ . Without loss of generality, we may assume that  $\frac{a_1}{1} \in P_p \setminus P_p^2$ . Note that we already have  $\frac{a_1}{1} \in p_p \setminus p_p^2$ . Now,  $\frac{P}{(a_1) \otimes_k K}$  is a prime ideal of  $\frac{A}{(a_1)} \otimes_k K$  with  $\frac{P}{(a_1) \otimes_k K} \cap \frac{A}{(a_1)} = \frac{p}{(a_1)}$ . By [24, Theorem 159], we obtain  $\text{embdim}\left(\left(\frac{A}{(a_1)}\right)_{\frac{p}{(a_1)}}\right) = n - 1$ . By induction, we get

$$\text{embdim}\left(\left(\frac{A}{(a_1)} \otimes_k K\right)_{\frac{P}{(a_1) \otimes_k K}}\right) = \text{embdim}\left(\left(\frac{A}{(a_1)}\right)_{\frac{p}{(a_1)}}\right).$$

We conclude, via [24, Theorem 159], to get

$$\text{embdim}(A \otimes_k K)_P = 1 + \text{embdim}\left(\left(\frac{A}{(a_1)} \otimes_k K\right)_{\frac{P}{(a_1) \otimes_k K}}\right) = n.$$

Moreover, observe that  $\left(\kappa_A(p) \otimes_k K\right)_{\frac{P_p}{p_p \otimes_k K}}$  is a field as  $P_p = (p \otimes_k K)_P$ . By Proposition 4.1, we have

$$\mu_P(pA_p) = \text{embdim}(A \otimes_k K)_P = \text{embdim}(A_p). \quad (12)$$

**Step 2.** Assume that  $B$  is an arbitrary  $k$ -algebra. Since  $P_p = (p \otimes_k B + A \otimes_k q)_P$ , then  $P(A_p \otimes_k B_q) = pA_p \otimes_k B_q + A_p \otimes_k qB_q$ , hence  $\left(\kappa_A(p) \otimes_k \kappa_B(q)\right)_{\frac{P(A_p \otimes_k B_q)}{pA_p \otimes_k B_q + A_p \otimes_k qB_q}}$  is an extension field of  $k$ . So, apply Proposition 4.1 twice to get

$$\text{embdim}(A \otimes_k B)_P = \mu_P(qB_q) + \mu_{\frac{P_q}{A \otimes_k qB_q}}(pA_p). \quad (13)$$

Further, notice that

$$\begin{aligned} \left(\frac{P_q}{A \otimes_k qB_q}\right)_{\frac{P_q}{A \otimes_k qB_q}} &= \frac{(P_q)_{P_q}}{(A \otimes_k qB_q)_{P_q}} = \frac{P_p}{(A \otimes_k q)_P} = \frac{(p \otimes_k B + A \otimes_k q)_P}{(A \otimes_k q)_P} \\ &= \left(\frac{p \otimes_k B + A \otimes_k q}{A \otimes_k q}\right)_{\frac{P}{A \otimes_k q}} \cong \left(p \otimes_k \frac{B}{q}\right)_{\frac{P}{A \otimes_k q}} = \left(p \otimes_k \kappa_B(q)\right)_{\frac{P_q}{A \otimes_k qB_q}}. \end{aligned}$$

Therefore, by (12), we get

$$\mu_{\frac{P_q}{A \otimes_k qB_q}}(pA_p) = \text{embdim}\left(\left(A \otimes_k \kappa_B(q)\right)_{\frac{P_q}{A \otimes_k qB_q}}\right) = \text{embdim}(A_p).$$

Similar arguments yield

$$\mu_{\frac{P_p}{pA_p \otimes_k B}}(qB_q) = \text{embdim}\left(\left(\kappa_A(p) \otimes_k B\right)_{\frac{P_p}{pA_p \otimes_k B}}\right) = \text{embdim}(B_q)$$

and, by the facts  $0 \leq \mu_P(pA_p) \leq \text{embdim}(A_p)$  and  $\mu_{\frac{P_p}{pA_p \otimes_k B}}(qB_q) \leq \mu_P(qB_q)$ , we obtain

$$\mu_P(pA_p) = \text{embdim}(A_p) \text{ and } \mu_P(qB_q) = \text{embdim}(B_q)$$

completing the proof of the lemma via (13).  $\square$

The second lemma will allow us to reduce our investigation to tensor products issued from finite extension fields.

**Lemma 4.4.** *Let  $K$  be an extension field of  $k$  and  $A$  a  $k$ -algebra such that  $K \otimes_k A$  is Noetherian. Let  $P$  be a prime ideal of  $K \otimes_k A$ . Then, there exists a finite extension field  $E$  of  $k$  contained in  $K$  such that*

$$\text{embdim}(K \otimes_k A)_P = \text{embdim}(F \otimes_k A)_Q$$

for each intermediate field  $F$  between  $E$  and  $K$  and  $Q := P \cap (F \otimes_k A)$ .

*Proof.* Let  $z_1, \dots, z_t \in K \otimes_k A$  such that  $P = (z_1, \dots, z_t) K \otimes_k A$ ; and for each  $i = 1, \dots, t$ , let  $z_i := \sum_{j=1}^{n_i} \alpha_{ij} \otimes_k a_j$  with  $\alpha_{ij} \in K$  and  $a_j \in A$ . Let  $E := k(\{\alpha_{ij} \mid i = 1, \dots, t; j = 1, \dots, n_i\})$  and  $Q := P \cap (E \otimes_k A)$ . Clearly,  $z_1, \dots, z_t \in Q$  and hence  $P = Q(K \otimes_k A) = K \otimes_E Q$ . Apply Lemma 4.3 to  $K \otimes_k A \cong K \otimes_E (E \otimes_k A)$  to obtain  $\text{embdim}(K \otimes_k A)_P = \text{embdim}(E \otimes_k A)_Q$ . Now, let  $F$  be an intermediate field between  $E$  and  $K$  and  $Q' := P \cap (F \otimes_k A)$ . Then

$$P = Q'(K \otimes_k A) = K \otimes_E Q' \quad (14)$$

since  $Q' \cap (E \otimes_k A) = Q$ . As above, Lemma 4.3 leads to the conclusion.  $\square$

Next, we give the proof of the main theorem.

*Proof of Theorem 4.2.* We proceed through three steps.

**Step 1.** Assume that  $K$  is an algebraic separable extension field of  $k$ . We claim that

$$P_P = (K \otimes_k p)_P. \quad (15)$$

Indeed, set  $S_p := \frac{A}{p} \setminus \{\bar{0}\}$ . The basic fact  $\frac{P}{K \otimes_k p} \cap \frac{A}{p} = (\bar{0})$  yields

$$\frac{(K \otimes_k A)_P}{(K \otimes_k p)_P} \cong \left( K \otimes_k \frac{A}{p} \right)_{\frac{P}{K \otimes_k p}} = \left( K \otimes_k \kappa_A(p) \right)_{S_p^{-1}(\frac{P}{K \otimes_k p})}$$

where  $K \otimes_k \kappa_A(p)$  is a zero-dimensional ring [30, Theorem 3.1], reduced [37, Chap. III, §15, Theorem 39], and hence von Neumann regular [24, Ex. 22, p. 64]. It follows that  $\left( K \otimes_k \kappa_A(p) \right)_{S_p^{-1}(\frac{P}{K \otimes_k p})}$  is a field. Consequently,  $(K \otimes_k p)_P = P_P$ , the unique maximal ideal of  $(K \otimes_k A)_P$ , proving our claim. By (15) and Lemma 4.3, we get  $\text{embdim}(K \otimes_k A)_P = \text{embdim}(A_p)$ .

**Step 2.** Assume that  $K$  is a finitely generated separable extension field of  $k$ . Let  $T = \{x_1, \dots, x_t\}$  be a finite separating transcendence base of  $K$  over  $k$ ; that is,  $K$  is algebraic separable over  $k(T) := k(x_1, \dots, x_t)$ . Let  $S := k[T] \setminus \{0\}$  and notice that

$$K \otimes_k A \cong K \otimes_{k(T)} (k(T) \otimes_k A) \cong K \otimes_{k(T)} S^{-1}A[T].$$

Let  $P \cap S^{-1}A[T] = S^{-1}P'$  for some prime ideal  $P'$  of  $A[T]$ . Note that  $P' \cap A = p$ . Then, we have

$$\begin{aligned} \text{embdim}(K \otimes_k A)_p &= \text{embdim}(K \otimes_{k(T)} S^{-1}A[T])_p \\ &= \text{embdim}(S^{-1}A[T]_{S^{-1}P'}), \text{ by Step 1} \\ &= \text{embdim}(A[T]_{P'}) \\ &= \text{embdim}(A_p) + \text{ht}\left(\frac{P'}{p[T]}\right), \text{ by Theorem 3.1.} \end{aligned}$$

Moreover, note that

$$\begin{aligned} K \otimes_k \frac{A}{p} &\cong K \otimes_{k(T)} \left( k(T) \otimes_k \frac{A}{p} \right) \\ &\cong K \otimes_{k(T)} \frac{S^{-1}A[T]}{S^{-1}p[T]} \end{aligned}$$

and

$$\frac{P}{K \otimes_k p} \cap \frac{S^{-1}A[T]}{S^{-1}p[T]} = \frac{S^{-1}P'}{S^{-1}p[T]}$$

as  $K \otimes_k p \cong K \otimes_{k(T)} S^{-1}p[T]$  so that  $(K \otimes_k p) \cap S^{-1}A[T] = S^{-1}p[T]$ . Therefore the integral extension  $\frac{S^{-1}A[T]}{S^{-1}p[T]} \hookrightarrow K \otimes_k \frac{A}{p}$  is flat and hence satisfies the Going-down property; that is,  $\text{ht}\left(\frac{P'}{p[T]}\right) = \text{ht}\left(\frac{S^{-1}P'}{S^{-1}p[T]}\right) = \text{ht}\left(\frac{P}{K \otimes_k p}\right)$ . It follows that  $\text{embdim}(K \otimes_k A)_p = \text{embdim}(A_p) + \text{ht}\left(\frac{P}{K \otimes_k p}\right)$ .

**Step 3.** Assume that  $K$  is an arbitrary separable extension field of  $k$ . Then, by Lemma 4.4, there exists a finite extension field  $E$  of  $k$  contained in  $K$  such that

$$\text{embdim}(K \otimes_k A)_p = \text{embdim}(E \otimes_k A)_Q$$

where  $Q := P \cap (E \otimes_k A)$ . Let  $\Omega$  denote the set of all intermediate fields between  $E$  and  $K$ . For each  $F \in \Omega$ , note that  $P = Q'(K \otimes_k A)$ , where  $Q' := P \cap (F \otimes_k A)$ , as seen in (14); and by Lemma 4.4 and Step 2, we obtain

$$\text{embdim}(K \otimes_k A)_p = \text{embdim}(F \otimes_k A)_{Q'} = \text{embdim}(A_p) + \text{ht}\left(\frac{Q'}{F \otimes_k p}\right). \quad (16)$$

Further, as the ring extension  $F \otimes_k \frac{A}{p} \hookrightarrow K \otimes_k \frac{A}{p}$  satisfies the Going-down property, we get

$$\text{ht}\left(\frac{Q'}{F \otimes_k p}\right) \leq \text{ht}\left(\frac{P}{K \otimes_k p}\right). \quad (17)$$

Next let  $K \otimes_k p \subseteq P_0 \subsetneq P_1 \subsetneq \dots \subsetneq P_n = P$  be a chain of distinct prime ideals of  $K \otimes_k A$  such that  $n := \text{ht}\left(\frac{P}{K \otimes_k p}\right)$ . Let  $t_i \in P_i \setminus P_{i-1}$  for each  $i = 1, \dots, n$ . One readily checks that there exists a finite extension field  $G$  of  $k$  contained in  $K$  such that, for each  $i = 1, \dots, n$ ,  $t_i \in G \otimes_k A$  and thus  $t_i \in Q'_i \setminus Q'_{i-1}$ , where  $Q'_i := P_i \cap (G \otimes_k A)$ . Let  $H := k(E, G) \in \Omega$  and  $Q_i := P_i \cap (H \otimes_k A)$  for each  $i = 1, \dots, n$ . Then  $t_i \in Q_i \setminus Q_{i-1}$  for each  $i = 1, \dots, n$ . Therefore, we get the following chain of distinct prime ideals in  $H \otimes_k A$

$$H \otimes_k p \subseteq Q_0 \subsetneq Q_1 \subsetneq \dots \subsetneq Q_n = Q' := P \cap (H \otimes_k A).$$

It follows that  $\text{ht}\left(\frac{Q'}{H \otimes_k p}\right) \geq n$  and then (17) yields  $\text{ht}\left(\frac{Q'}{F \otimes_k p}\right) = \text{ht}\left(\frac{P}{K \otimes_k p}\right)$ . Further,  $K \otimes_k \kappa_A(p)$  is regular since  $K$  is separable over  $k$  [18, Lemma 6.7.4.1]. Consequently, by (16), we get

$$\begin{aligned} \text{embdim}(K \otimes_k A)_p &= \text{embdim}(A_p) + \text{ht}\left(\frac{P}{K \otimes_k p}\right) \\ &= \text{embdim}(A_p) + \text{embdim}\left(\left(K \otimes_k \kappa_A(p)\right)_{\frac{P_p}{K \otimes_k p A_p}}\right) \end{aligned}$$

completing the proof of the theorem.

As a direct application of Theorem 4.2, we obtain the next corollary on the (embedding) codimension which extends Grothendieck's result on the transfer of regularity to tensor products issued from finite extension fields [18, Lemma 6.7.4.1]. See also [8].

**Corollary 4.5.** *Let  $K$  be a separable extension field of  $k$  and  $A$  a  $k$ -algebra such that  $K \otimes_k A$  is Noetherian. Let  $P$  be a prime ideal of  $K \otimes_k A$  with  $p := P \cap A$ . Then:*

$$\text{codim}(K \otimes_k A)_p = \text{codim}(A_p).$$

*In particular,  $K \otimes_k A$  is regular if and only if  $A$  is regular.*

*Proof.* Combine Theorem 4.2 and (11). □

## 5. EMBEDDING DIMENSION AND CODIMENSION OF TENSOR PRODUCTS OF ALGEBRAS WITH SEPARABLE RESIDUE FIELDS

This section deals with a more general setting (than in Section 4); namely, we compute the embedding dimension of localizations of the tensor product of two  $k$ -algebras  $A$  and  $B$  at prime ideals  $P$  such that the residue field  $\kappa_B(P \cap B)$  is a separable extension of  $k$ . The main result establishes an analogue for an extended form of the "special chain theorem" for the Krull dimension which asserts that

$$\begin{aligned} \dim(A \otimes_k B)_P &= \dim(A_p) + \dim(B_q) + \text{ht}\left(\frac{P}{p \otimes_k B + A \otimes_k q}\right) \\ &= \dim(A_p) + \dim(B_q) + \dim\left(\left(\kappa_A(p) \otimes_k \kappa_B(q)\right)_{\frac{P(A_p \otimes_k B_q)}{p A_p \otimes_k B_q + A_p \otimes_k q B_q}}\right). \end{aligned} \quad (18)$$

As an application, we formulate the (embedding) codimension of these constructions with direct consequence on the transfer or defect of regularity.

Here is the main result of this section.

**Theorem 5.1.** *Let  $A$  and  $B$  be two  $k$ -algebras such that  $A \otimes_k B$  is Noetherian and let  $P$  be a prime ideal of  $A \otimes_k B$  with  $p := P \cap A$  and  $q := P \cap B$ . Assume  $\kappa_B(q)$  is separable over  $k$ . Then:*

$$\begin{aligned} \text{embdim}(A \otimes_k B)_P &= \text{embdim}(A_p) + \text{embdim}(B_q) + \text{ht}\left(\frac{P}{p \otimes_k B + A \otimes_k q}\right) \\ &= \text{embdim}(A_p) + \text{embdim}(B_q) \\ &\quad + \text{embdim}\left(\left(\kappa_A(p) \otimes_k \kappa_B(q)\right)_{\frac{P(A_p \otimes_k B_q)}{p A_p \otimes_k B_q + A_p \otimes_k q B_q}}\right) \end{aligned}$$

*Proof.* Notice first that, as  $\kappa_B(q)$  is separable over  $k$ ,  $\kappa_A(p) \otimes_k \kappa_B(q)$  is a regular ring and hence

$$\begin{aligned} \text{embdim} \left( \left( \kappa_A(p) \otimes_k \kappa_B(q) \right)_{\frac{P(A_p \otimes_k B_q)}{pA_p \otimes_k B_q + A_p \otimes_k qB_q}} \right) &= \text{ht} \left( \frac{P(A_p \otimes_k B_q)}{pA_p \otimes_k B_q + A_p \otimes_k qB_q} \right) \\ &= \text{ht} \left( \frac{P}{p \otimes_k B + A \otimes_k q} \right). \end{aligned}$$

So, we only need to prove the first equality in the theorem and, without loss of generality, we may assume that  $(A, \mathfrak{n})$  and  $(B, \mathfrak{m})$  are local  $k$ -algebras such that  $A \otimes_k B$  is Noetherian,  $\frac{B}{\mathfrak{m}}$  is a separable extension field of  $k$ , and  $P$  is a prime ideal of  $A \otimes_k B$  with  $P \cap A = \mathfrak{n}$  and  $P \cap B = \mathfrak{m}$ . Similar arguments used in the proof of Lemma 4.4 show that there exists a finite extension field  $K$  of  $k$  contained in  $\frac{B}{\mathfrak{m}}$  such that

$$\frac{P}{A \otimes_k \mathfrak{m}} = Q \left( A \otimes_k \frac{B}{\mathfrak{m}} \right) \cong Q \otimes_K \frac{B}{\mathfrak{m}}$$

where  $Q := \frac{P}{A \otimes_k \mathfrak{m}} \cap (A \otimes_k K)$ . Since  $\frac{B}{\mathfrak{m}}$  is separable over  $k$  and  $K$  is a finitely generated intermediate field, then  $K$  is separably generated over  $k$  (cf. [21, Chap. VI, Theorem 2.10 & Definition 2.11]). Let  $t$  denote the transcendence degree of  $K$  over  $k$  and let  $c_1, \dots, c_t \in B$  such that  $\{\overline{c_1}, \dots, \overline{c_t}\}$  is a separating transcendence base of  $K$  over  $k$ ; i.e.,  $K$  is separable algebraic over  $\Omega := k(\overline{c_1}, \dots, \overline{c_t})$ . Also  $c_1, \dots, c_t$  are algebraically independent over  $k$  with

$$\mathfrak{m} \cap k[c_1, \dots, c_t] = (0). \quad (19)$$

So one may view  $A \otimes_k k[c_1, \dots, c_t] \cong A[c_1, \dots, c_t]$  as a polynomial ring in  $t$  indeterminates over  $A$ . Set  $S := k[c_1, \dots, c_t] \setminus \{0\}$ ;  $k(t) := k(c_1, \dots, c_t)$ ;  $A[t] := A[c_1, \dots, c_t]$ . Then, we have

$$P \cap S = \mathfrak{m} \cap S = \emptyset \text{ and } A \otimes_k S^{-1} B \cong S^{-1} A[t] \otimes_{k(t)} S^{-1} B. \quad (20)$$

Next, let  $T := \frac{P}{A \otimes_k \mathfrak{m}} \cap (A \otimes_k \Omega) = Q \cap (A \otimes_k \Omega)$  and consider the following canonical isomorphisms of  $k$ -algebras  $\theta_1 : A \otimes_k \frac{S^{-1} B}{S^{-1} \mathfrak{m}} \longrightarrow (A \otimes_k k(t)) \otimes_{k(t)} \frac{S^{-1} B}{S^{-1} \mathfrak{m}}$  and  $\theta_2 : A \otimes_k \frac{B}{\mathfrak{m}} \longrightarrow (A \otimes_k \Omega) \otimes_{\Omega} \frac{B}{\mathfrak{m}}$ . As  $A \otimes_k K \cong (A \otimes_k \Omega) \otimes_{\Omega} K$ , by (15) we obtain  $Q_Q = (T \otimes_{\Omega} K)_Q = T(A \otimes_k K)_Q$  and hence

$$\begin{aligned} \left( \frac{P}{A \otimes_k \mathfrak{m}} \right)_{\frac{P}{A \otimes_k \mathfrak{m}}} &= Q \left( A \otimes_k \frac{B}{\mathfrak{m}} \right)_{\frac{P}{A \otimes_k \mathfrak{m}}} = Q_Q \left( A \otimes_k \frac{B}{\mathfrak{m}} \right)_{\left( \frac{P}{A \otimes_k \mathfrak{m}} \right)_Q} \\ &= T(A \otimes_k K)_Q \left( A \otimes_k \frac{B}{\mathfrak{m}} \right)_{\left( \frac{P}{A \otimes_k \mathfrak{m}} \right)_Q} \\ &= T(A \otimes_k K) \left( A \otimes_k \frac{B}{\mathfrak{m}} \right)_{\frac{P}{A \otimes_k \mathfrak{m}}} = T \left( A \otimes_k \frac{B}{\mathfrak{m}} \right)_{\frac{P}{A \otimes_k \mathfrak{m}}} \\ &= \left( \theta_2^{-1} \left( \theta_1 \left( T \left( A \otimes_k \frac{B}{\mathfrak{m}} \right) \right) \right) \right)_{\frac{P}{A \otimes_k \mathfrak{m}}} = \left( \theta_2^{-1} \left( T \otimes_{\Omega} \frac{B}{\mathfrak{m}} \right) \right)_{\frac{P}{A \otimes_k \mathfrak{m}}}. \end{aligned} \quad (21)$$

Moreover, on account of (19) and by considering the natural surjective homomorphism of  $k$ -algebras  $k[c_1, \dots, c_t] \xrightarrow{\varphi} k[\overline{c_1}, \dots, \overline{c_t}]$  defined by  $\varphi(c_i) = \overline{c_i}$  for each  $i = 1, \dots, t$ ,



we get  $k[c_1, \dots, c_t] \xrightarrow{\varphi} k[\overline{c_1}, \dots, \overline{c_t}]$  inducing the following natural isomorphism of extension fields  $\phi := S^{-1}\varphi : k(t) \longrightarrow k(\overline{c_1}, \dots, \overline{c_t}) = \Omega$ . Then,  $\phi$  induces a structure of  $k(t)$ -algebras on  $\Omega$  and thus on  $\frac{B}{\mathfrak{m}}$ . We adopt a second structure of  $k(t)$ -algebras on  $\frac{B}{\mathfrak{m}}$ , inherited from the canonical injection  $k(t) \xhookrightarrow{i} S^{-1}B$ . Indeed, consider the following  $k$ -algebra homomorphisms  $k(t) \xrightarrow{\bar{i}} \frac{S^{-1}B}{S^{-1}\mathfrak{m}} \xrightarrow{\gamma} \frac{B}{\mathfrak{m}}$  defined by  $\bar{i}(\alpha) = \overline{\alpha}$  for each  $\alpha \in k(t)$ , and where  $\gamma$  is the isomorphism of  $k$ -algebras defined by  $\gamma\left(\frac{\overline{b}}{s}\right) = \frac{\overline{b}}{s}$  for each  $b \in B$  and each  $s \in S$ . It is easy to see that these two structures of  $k(t)$ -algebras coincide on  $\frac{B}{\mathfrak{m}}$ . This is due to the commutativity of the following diagram of homomorphisms of  $k$ -algebras

$$\begin{array}{ccc} k(t) & \xrightarrow{\bar{i}} & \frac{S^{-1}B}{S^{-1}\mathfrak{m}} \\ & \searrow \phi & \downarrow \gamma \\ & & \frac{B}{\mathfrak{m}} \end{array}$$

since, for each  $\alpha := \frac{f}{s} \in k(t)$  with  $f \in k[c_1, \dots, c_t]$  and  $s \in S$ , we have

$$(\gamma \circ \bar{i})(\alpha) = \gamma(\overline{\alpha}) = \frac{\overline{f}}{s} = \frac{\varphi(f)}{\varphi(s)} = \phi(\alpha).$$

Now, consider the following isomorphism of  $k$ -algebras

$$\psi := \theta_2 \circ (1_A \otimes_k \gamma) \circ \theta_1^{-1} : (A \otimes_k k(t)) \otimes_{k(t)} \frac{S^{-1}B}{S^{-1}\mathfrak{m}} \longrightarrow (A \otimes_k \Omega) \otimes_{\Omega} \frac{B}{\mathfrak{m}}$$

where, for each  $a \in A$ ,  $\alpha \in k(t)$ ,  $b \in B$ , and  $s \in S$ , we have

$$\begin{aligned} \psi\left((a \otimes_k \alpha) \otimes_{k(t)} \frac{\overline{b}}{s}\right) &= \theta_2\left((1_A \otimes_k \gamma)\left(a \otimes_k \overline{\alpha \frac{\overline{b}}{s}}\right)\right) = \theta_2\left(a \otimes_k \gamma\left(\overline{\alpha \frac{\overline{b}}{s}}\right)\right) \\ &= \theta_2\left(a \otimes_k \left((\gamma \circ \bar{i})(\alpha) \gamma\left(\frac{\overline{b}}{s}\right)\right)\right) = \theta_2\left(a \otimes_k \left(\phi(\alpha) \gamma\left(\frac{\overline{b}}{s}\right)\right)\right) \\ &= (a \otimes_k \overline{1}) \otimes_{\Omega} \phi(\alpha) \gamma\left(\frac{\overline{b}}{s}\right) = (a \otimes_k \phi(\alpha)) \otimes_{\Omega} \gamma\left(\frac{\overline{b}}{s}\right) \\ &= (1_A \otimes_k \phi)(a \otimes_k \alpha) \otimes_{\Omega} \gamma\left(\frac{\overline{b}}{s}\right). \end{aligned}$$

Next, let  $\delta : A \otimes_k S^{-1}B \longrightarrow S^{-1}A[t] \otimes_{k(t)} S^{-1}B$  denote the canonical isomorphism of  $k$ -algebras mentioned in (20) and let  $S^{-1}H := S^{-1}P \cap S^{-1}A[t]$  where  $H$  is a prime ideal of  $A[t]$  with  $H \cap S = \emptyset$ . Therefore

$$\begin{aligned} \psi\left(S^{-1}H \otimes_{k(t)} \frac{S^{-1}B}{S^{-1}\mathfrak{m}}\right) &= (1_A \otimes_k \phi)(S^{-1}H) \otimes_{\Omega} \gamma\left(\frac{S^{-1}B}{S^{-1}\mathfrak{m}}\right) \\ &= (1_A \otimes_k \phi)(S^{-1}H) \otimes_{\Omega} \frac{B}{\mathfrak{m}}. \end{aligned} \tag{22}$$

**Claim:**  $\delta(S^{-1}P)_{\delta(S^{-1}P)} = (S^{-1}H \otimes_{k(t)} S^{-1}B + S^{-1}A[t] \otimes_{k(t)} S^{-1}\mathfrak{m})_{\delta(S^{-1}P)}.$

Indeed, consider the following commutative diagram (as  $\phi = \gamma \circ \bar{i}$ )

$$\begin{array}{ccc} S^{-1}(A \otimes_k B) = A \otimes_k S^{-1}B & \xrightarrow{1_A \otimes_k (\gamma \circ \pi)} & A \otimes_k \frac{B}{\mathfrak{m}} \\ 1_A \otimes_k i \uparrow & & \uparrow \\ A \otimes_k k(t) & \xrightarrow{1_A \otimes_k \phi} & A \otimes_k \Omega \end{array}$$

where  $\pi : S^{-1}B \rightarrow \frac{S^{-1}B}{S^{-1}\mathfrak{m}}$  denotes the canonical surjection (with  $\pi \circ i = \bar{i}$ ) and the vertical maps are the canonical injections. Also, it is worth noting that  $1_A \otimes_k \phi$  is an isomorphism of  $k$ -algebras. Hence

$$\begin{aligned} T &= \frac{P}{A \otimes_k \mathfrak{m}} \cap (A \otimes_k \Omega) = (1_A \otimes_k \phi) \left( \left( (1_A \otimes_k (\gamma \circ \pi))^{-1} \left( \frac{P}{A \otimes_k \mathfrak{m}} \right) \right) \cap (A \otimes_k k(t)) \right) \\ &= (1_A \otimes_k \phi) (S^{-1}P \cap (A \otimes_k k(t))) = (1_A \otimes_k \phi) (S^{-1}P \cap S^{-1}A[t]) \\ &= (1_A \otimes_k \phi) (S^{-1}H). \end{aligned} \tag{23}$$

It follows, via (21), (23), and (22), that

$$\begin{aligned} \left( \frac{P}{A \otimes_k \mathfrak{m}} \right)_{\frac{P}{A \otimes_k \mathfrak{m}}} &= \theta_2^{-1} \left( T \otimes_{\Omega} \frac{B}{\mathfrak{m}} \right)_{\frac{P}{A \otimes_k \mathfrak{m}}} \\ &= \theta_2^{-1} \left( (1_A \otimes_k \phi) (S^{-1}H) \otimes_{\Omega} \frac{B}{\mathfrak{m}} \right)_{\frac{P}{A \otimes_k \mathfrak{m}}} \\ &= \theta_2^{-1} \left( \psi \left( S^{-1}H \otimes_{k(t)} \frac{S^{-1}B}{S^{-1}\mathfrak{m}} \right) \right)_{\frac{P}{A \otimes_k \mathfrak{m}}} \\ &= (1_A \otimes_k \gamma) \left( \theta_1^{-1} \left( S^{-1}H \otimes_{k(t)} \frac{S^{-1}B}{S^{-1}\mathfrak{m}} \right) \right)_{\frac{P}{A \otimes_k \mathfrak{m}}}. \end{aligned}$$

Further, notice that  $\frac{P}{A \otimes_k \mathfrak{m}} = (1_A \otimes_k \gamma) \left( \frac{S^{-1}P}{A \otimes_k S^{-1}\mathfrak{m}} \right)$ . Then the isomorphism  $1_A \otimes_k \gamma$  yields the canonical isomorphism of local  $k$ -algebras

$$\begin{aligned} (1_A \otimes_k \gamma)_P : \left( A \otimes_k \frac{S^{-1}B}{S^{-1}\mathfrak{m}} \right)_{\frac{S^{-1}P}{A \otimes_k S^{-1}\mathfrak{m}}} &\longrightarrow \left( A \otimes_k \frac{B}{\mathfrak{m}} \right)_{\frac{P}{A \otimes_k \mathfrak{m}}} \text{ with} \\ (1_A \otimes_k \gamma)_P \left( \left( \frac{S^{-1}P}{A \otimes_k S^{-1}\mathfrak{m}} \right)_{\frac{S^{-1}P}{A \otimes_k S^{-1}\mathfrak{m}}} \right) &= \left( \frac{P}{A \otimes_k \mathfrak{m}} \right)_{\frac{P}{A \otimes_k \mathfrak{m}}} \\ &= (1_A \otimes_k \gamma)_P \left( \theta_1^{-1} \left( S^{-1}H \otimes_{k(t)} \frac{S^{-1}B}{S^{-1}\mathfrak{m}} \right)_{\frac{S^{-1}P}{A \otimes_k S^{-1}\mathfrak{m}}} \right). \end{aligned}$$

Therefore

$$\theta_1^{-1} \left( S^{-1}H \otimes_{k(t)} \frac{S^{-1}B}{S^{-1}\mathfrak{m}} \right)_{\frac{S^{-1}P}{A \otimes_k S^{-1}\mathfrak{m}}} = \left( \frac{S^{-1}P}{A \otimes_k S^{-1}\mathfrak{m}} \right)_{\frac{S^{-1}P}{A \otimes_k S^{-1}\mathfrak{m}}}. \tag{24}$$

Moreover, consider the following commutative diagram

$$\begin{array}{ccc} A \otimes_k S^{-1}B & \xrightarrow{\delta} & S^{-1}A[t] \otimes_{k(t)} S^{-1}B \\ \pi_1 \downarrow & & \downarrow \pi_2 \\ A \otimes_k \frac{S^{-1}B}{S^{-1}\mathfrak{m}} & \xrightarrow{\theta_1} & S^{-1}A[t] \otimes_{k(t)} \frac{S^{-1}B}{S^{-1}\mathfrak{m}} \end{array}$$

where  $\pi_1 = 1_A \otimes_k \pi$  and  $\pi_2 = 1_{S^{-1}A[t]} \otimes_k \pi$  are the canonical surjective homomorphisms of  $k$ -algebras. Hence

$$\begin{aligned} \pi_1^{-1} \left( \theta_1^{-1} \left( S^{-1}H \otimes_{k(t)} \frac{S^{-1}B}{S^{-1}\mathfrak{m}} \right) \right) &= (\theta_1 \circ \pi_1)^{-1} \left( S^{-1}H \otimes_{k(t)} \frac{S^{-1}B}{S^{-1}\mathfrak{m}} \right) \\ &= (\pi_2 \circ \delta)^{-1} \left( S^{-1}H \otimes_{k(t)} \frac{S^{-1}B}{S^{-1}\mathfrak{m}} \right) \\ &= \delta^{-1} \left( \pi_2^{-1} \left( S^{-1}H \otimes_{k(t)} \frac{S^{-1}B}{S^{-1}\mathfrak{m}} \right) \right) \\ &= \delta^{-1} \left( S^{-1}H \otimes_{k(t)} S^{-1}B + S^{-1}A[t] \otimes_{k(t)} S^{-1}\mathfrak{m} \right) \end{aligned}$$

so that

$$\begin{aligned} \theta_1^{-1} \left( S^{-1}H \otimes_{k(t)} \frac{S^{-1}B}{S^{-1}\mathfrak{m}} \right) &= \pi_1 \left( \delta^{-1} \left( S^{-1}H \otimes_{k(t)} S^{-1}B + S^{-1}A[t] \otimes_{k(t)} S^{-1}\mathfrak{m} \right) \right) \\ &= \frac{\delta^{-1} \left( S^{-1}H \otimes_{k(t)} S^{-1}B + S^{-1}A[t] \otimes_{k(t)} S^{-1}\mathfrak{m} \right)}{A \otimes_k S^{-1}\mathfrak{m}}. \end{aligned}$$

It follows, via (24), that

$$\begin{aligned} \frac{S^{-1}P_{S^{-1}P}}{(A \otimes_k S^{-1}\mathfrak{m})_{S^{-1}P}} &= \left( \frac{S^{-1}P}{A \otimes_k S^{-1}\mathfrak{m}} \right)_{\frac{S^{-1}P}{A \otimes_k S^{-1}\mathfrak{m}}} = \theta_1^{-1} \left( S^{-1}H \otimes_{k(t)} \frac{S^{-1}B}{S^{-1}\mathfrak{m}} \right)_{\frac{S^{-1}P}{A \otimes_k S^{-1}\mathfrak{m}}} \\ &= \left( \frac{\delta^{-1} \left( S^{-1}H \otimes_{k(t)} S^{-1}B + S^{-1}A[t] \otimes_{k(t)} S^{-1}\mathfrak{m} \right)}{A \otimes_k S^{-1}\mathfrak{m}} \right)_{\frac{S^{-1}P}{A \otimes_k S^{-1}\mathfrak{m}}} \\ &= \frac{\delta^{-1} \left( S^{-1}H \otimes_{k(t)} S^{-1}B + S^{-1}A[t] \otimes_{k(t)} S^{-1}\mathfrak{m} \right)_{S^{-1}P}}{(A \otimes_k S^{-1}\mathfrak{m})_{S^{-1}P}} \end{aligned}$$

and thus  $S^{-1}P_{S^{-1}P} = \delta^{-1} \left( S^{-1}H \otimes_{k(t)} S^{-1}B + S^{-1}A[t] \otimes_{k(t)} S^{-1}\mathfrak{m} \right)_{S^{-1}P}$ . Also, note that the isomorphism of  $k$ -algebras  $\delta$  induces the isomorphism of local  $k$ -algebras  $\delta_P : (A \otimes_k S^{-1}B)_{S^{-1}P} \longrightarrow (S^{-1}A[t] \otimes_{k(t)} S^{-1}B)_{\delta(S^{-1}P)}$ . Hence

$$\begin{aligned} \delta_P^{-1} \left( \delta(S^{-1}P)_{\delta(S^{-1}P)} \right) &= S^{-1}P_{S^{-1}P} \\ &= \delta_P^{-1} \left( \left( S^{-1}H \otimes_{k(t)} S^{-1}B + S^{-1}A[t] \otimes_{k(t)} S^{-1}\mathfrak{m} \right)_{\delta(S^{-1}P)} \right) \end{aligned}$$

so that  $\delta(S^{-1}P)_{\delta(S^{-1}P)} = \left( S^{-1}H \otimes_{k(t)} S^{-1}B + S^{-1}A[t] \otimes_{k(t)} S^{-1}\mathfrak{m} \right)_{\delta(S^{-1}P)}$  proving the claim.

It follows, by Lemma 4.3 applied to  $S^{-1}A[t] \otimes_{k(t)} S^{-1}B$ , that

$$\mu_{\delta(S^{-1}P)}(S^{-1}\mathfrak{m}S^{-1}B_{S^{-1}\mathfrak{m}}) = \text{embdim}(S^{-1}B_{S^{-1}\mathfrak{m}}) = \text{embdim}(B)$$

so that, by Proposition 4.1, we have

$$\begin{aligned}
\text{embdim}(A \otimes_k B)_P &= \text{embdim}\left((A \otimes_k S^{-1}B)_{S^{-1}P}\right) = \text{embdim}\left((S^{-1}A[t] \otimes_{k(t)} S^{-1}B)_{\delta(S^{-1}P)}\right) \\
&= \mu_{\delta(S^{-1}P)}(S^{-1}\mathfrak{m}S^{-1}B_{S^{-1}\mathfrak{m}}) + \text{embdim}\left(\left(S^{-1}A[t] \otimes_{k(t)} \frac{S^{-1}B}{S^{-1}\mathfrak{m}}\right)_{\frac{\delta(S^{-1}P)}{S^{-1}A[t] \otimes_{k(t)} S^{-1}\mathfrak{m}}}\right) \\
&= \text{embdim}(B) + \text{embdim}\left(\left(S^{-1}A[t] \otimes_{k(t)} \frac{S^{-1}B}{S^{-1}\mathfrak{m}}\right)_{\frac{\delta(S^{-1}P)}{S^{-1}A[t] \otimes_{k(t)} S^{-1}\mathfrak{m}}}\right) \\
&= \text{embdim}(B) + \text{embdim}\left(\left(A \otimes_k \frac{S^{-1}B}{S^{-1}\mathfrak{m}}\right)_{\frac{S^{-1}P}{A \otimes_k S^{-1}\mathfrak{m}}}\right).
\end{aligned}$$

Finally, as  $\frac{S^{-1}B}{S^{-1}\mathfrak{m}} \cong \frac{B}{\mathfrak{m}}$  is a separable extension field of  $k$ , we get, by Theorem 4.2, that

$$\begin{aligned}
\text{embdim}(A \otimes_k B)_P &= \text{embdim}(A) + \text{embdim}(B) + \text{ht}\left(\frac{S^{-1}P/(A \otimes_k S^{-1}\mathfrak{m})}{\mathfrak{n} \otimes_k (S^{-1}B/S^{-1}\mathfrak{m})}\right) \\
&= \text{embdim}(A) + \text{embdim}(B) + \text{ht}\left(\frac{S^{-1}P}{\mathfrak{n} \otimes_k S^{-1}B + A \otimes_k S^{-1}\mathfrak{m}}\right) \\
&= \text{embdim}(A) + \text{embdim}(B) + \text{ht}\left(\frac{P}{\mathfrak{n} \otimes_k B + A \otimes_k \mathfrak{m}}\right)
\end{aligned}$$

completing the proof of the theorem.  $\square$

As a direct application of Theorem 5.1, we obtain the next corollary on the (embedding) codimension which recovers known results on the transfer of regularity to tensor products over perfect fields [33, Theorem 6(c)] and, more generally, to tensor products issued from residually separable extension fields [8, Theorem 2.11]. Recall that a  $k$ -algebra  $R$  is said to be residually separable, if  $\kappa_R(p)$  is separable over  $k$  for each prime ideal  $p$  of  $R$ .

**Corollary 5.2.** *Let  $A$  and  $B$  be two  $k$ -algebras such that  $A \otimes_k B$  is Noetherian and let  $P$  be a prime ideal of  $A \otimes_k B$  with  $p := P \cap A$  and  $q := P \cap B$ . Assume  $\kappa_B(q)$  is separable over  $k$ . Then:*

$$\text{codim}(A \otimes_k B)_P = \text{codim}(A_p) + \text{codim}(B_q).$$

*Proof.* Combine Theorem 5.1 and (18).  $\square$

Note that if  $k$  is perfect, then every  $k$ -algebra is residually separable. Now, if  $k$  is an arbitrary field, one can easily provide original examples of residually separable  $k$ -algebras through localizations of polynomial rings or pullbacks [2, 13]. For instance, let  $X$  be an indeterminate over  $k$  and  $K \subseteq L$  two separable extensions of  $k$ . Then, the one-dimensional local  $k$ -algebras  $R := K + XL[X]_{(X)} \subseteq S := L[X]_{(X)}$  are residually separable since the extensions  $k \subseteq \kappa_R(XL[X]_{(X)}) = K \subseteq \kappa_S(XL[X]_{(X)}) = L \subseteq \kappa_R(0) = \kappa_S(0) = L(X)$  are separable over  $k$  by Mac Lane's Criterion and transitivity of separability. Also, similar arguments show that the two-dimensional local  $k$ -algebra  $R' := R + YL(X)[Y]_{(Y)}$  is residually separable, where  $Y$  is an indeterminate over  $k$ . Therefore, one may reiterate the same process to build residually separable  $k$ -algebras of arbitrary Krull dimension.

**Corollary 5.3.** *Let  $A$  be a finitely generated  $k$ -algebra and  $B$  a residually separable  $k$ -algebra. Let  $P$  be a prime ideal of  $A \otimes_k B$  with  $p := P \cap A$  and  $q := P \cap B$ . Then:*

$$\text{codim}(A \otimes_k B)_P = \text{codim}(A_p) + \text{codim}(B_q).$$

*In particular,  $A \otimes_k B$  is regular if and only if so are  $A$  and  $B$ .*

**Corollary 5.4.** *Let  $k$  be an algebraically closed field,  $A$  a finitely generated  $k$ -algebra,  $p$  a maximal ideal of  $A$ , and  $B$  an arbitrary  $k$ -algebra. Let  $P$  be a prime ideal of  $A \otimes_k B$  such that  $P \cap A = p$  and set  $q := P \cap B$ . Then:*

$$\text{codim}(A \otimes_k B)_P = \text{codim}(A_p) + \text{codim}(B_q).$$

#### REFERENCES

- [1] D. F. Anderson, A. Bouvier, D. E. Dobbs, M. Fontana, and S. Kabbaj, On Jaffard domains, *Expo. Math.* **6** (2) (1988) 145–175.
- [2] E. Bastida and R. Gilmer, Overrings and divisorial ideals of rings of the form  $D + M$ , *Michigan Math. J.* **20** (1973) 79–95.
- [3] S. Bouchiba, D. E. Dobbs, and S. Kabbaj, On the prime ideal structure of tensor products of algebras, *J. Pure Appl. Algebra* **176** (2002) 89–112.
- [4] S. Bouchiba, J. Conde-Lago, and J. Majadas, Cohen-Macaulay, Gorenstein, complete intersection and regular defect for the tensor product of algebras, Preprint (arXiv:1512.02804)
- [5] S. Bouchiba, F. Girolami and S. Kabbaj, The dimension of tensor products of AF-rings, pp. 141–154, *Lecture Notes in Pure Appl. Math.*, Vol. 185, Dekker, New York, 1997.
- [6] S. Bouchiba, F. Girolami and S. Kabbaj, The dimension of tensor products of  $k$ -algebras arising from pullbacks, *J. Pure Appl. Algebra* **137** (1999) 125–138.
- [7] S. Bouchiba and S. Kabbaj, Tensor products of Cohen-Macaulay rings. Solution to a problem of Grothendieck, *J. Algebra* **252** (2002) 65–73.
- [8] S. Bouchiba and S. Kabbaj, Regularity of tensor products of  $k$ -algebras, *Math. Scand.* **115** (1) (2014) 5–19.
- [9] N. Bourbaki, *Algèbre*, Chapitres 4–7, Masson, Paris, 1981.
- [10] N. Bourbaki, *Algèbre Commutative*, Chapitres 8–9, Masson, Paris, 1981.
- [11] A. Bouvier, D. E. Dobbs, and M. Fontana, Universally catenarian integral domains, *Adv. in Math.* **72** (2) (1988) 211–238.
- [12] J. Brewer, P. Montgomery, E. Rutter and W. Heinzer, Krull dimension of polynomial rings, pp. 26–45, *Lecture Notes in Math.*, Vol. 311, Springer, Berlin, 1973.
- [13] J. W. Brewer and E. A. Rutter,  $D + M$  constructions with general overrings, *Michigan Math. J.* **23** (1976) 33–42.
- [14] W. Bruns and J. Herzog, *Cohen-Macaulay rings*, Cambridge University Press, Cambridge, 1993.
- [15] D. Eisenbud, *Commutative algebra with a view toward algebraic geometry*, GTM, vol. 150, Springer-Verlag, New York, 1995.
- [16] D. Ferrand, Monomorphismes et morphismes absolument plats, *Bull. Soc. Math. France* **100** (1972) 97–128.
- [17] M. Fontana and S. Kabbaj, Essential domains and two conjectures in dimension theory, *Proc. Amer. Math. Soc.* **132** (9) (2004) 2529–2535.
- [18] A. Grothendieck, *Eléments de géométrie algébrique*, Institut des Hautes Etudes Sci. Publ. Math. No. 24, Bures-sur-yvette, 1965.
- [19] H. Haghighi, M. Tousi, and S. Yassemi, Tensor product of algebras over a field, in: “Commutative algebra. Noetherian and non-Noetherian perspectives,” pp. 181–202, Springer, New York, 2011.
- [20] C. Huneke and D. A. Jorgensen, Symmetry in the vanishing of Ext over Gorenstein rings, *Math. Scand.* **93** (2) (2003) 161–184.
- [21] T. Hungerford, *Algebra*, Springer-Verlag, New York, 1974.
- [22] P. Jaffard, Théorie de la dimension dans les anneaux de polynômes, *Mém. Sc. Math.*, 146, Gauthier-Villars, Paris (1960).
- [23] D. A. Jorgensen, On tensor products of rings and extension conjectures, *J. Commut. Algebra* **1** (4) (2009) 635–646.
- [24] I. Kaplansky, *Commutative rings*, University of Chicago Press, Chicago, 1974.

- [25] J. Majadas, On tensor products of complete intersections, *Bull. Lond. Math. Soc.* **45** (6) (2013) 1281–1284.
- [26] H. Matsumura, *Commutative ring theory*, Cambridge University Press, Cambridge, 1986.
- [27] M. Nagata, *Local rings*, Robert E. Krieger Publishing Co., Huntington, N.Y., 1975.
- [28] J. J. Rotman, *An introduction to homological algebra*, Second edition, Universitext, Springer, New York, 2009.
- [29] R.Y. Sharp, Simplifications in the theory of tensor products of field extensions, *J. London Math. Soc.* **15** (1977) 48–50.
- [30] R.Y. Sharp, The dimension of the tensor product of two field extensions, *Bull. London Math. Soc.* **9** (1977) 42–48.
- [31] R.Y. Sharp, The effect on associated prime ideals produced by an extension of the base field, *Math. Scand.* **38** (1976) 43–52.
- [32] M. E. Sweedler, When is the tensor product of algebras local? *Proc. Amer. Math. Soc.* **48** (1975) 8–10.
- [33] M. Tousi and S. Yassemi, Tensor products of some special rings, *J. Algebra* **268** (2003) 672–676.
- [34] P. Vamos, On the minimal prime ideals of a tensor product of two fields, *Math. Proc. Camb. Phil. Soc.* **84** (1978) 25–35.
- [35] A.R. Wadsworth, The Krull dimension of tensor products of commutative algebras over a field, *J. London Math. Soc.* **19** (1979) 391–401.
- [36] K. Watanabe, T. Ishikawa, S. Tachibana, and K. Otsuka, On tensor products of Gorenstein rings, *J. Math. Kyoto Univ.* **9** (1969) 413–423.
- [37] O. Zariski and P. Samuel, *Commutative algebra*, Vol. I, Van Nostrand, Princeton, 1960.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MEKNES, MEKNES 50000, MOROCCO  
*E-mail address:* bouchiba@fs-umi.ac.ma

DEPARTMENT OF MATHEMATICS AND STATISTICS, KING FAHD UNIVERSITY OF PETROLEUM & MINERALS (KFUPM), DHAHRAN 31261, KSA  
*E-mail address:* kabbaj@kfupm.edu.sa